Super-resolution and sensor calibration in imaging

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Outline

1. Super-resolution
   - Resolution in imaging
   - Super-resolution limit and min-max error
   - Super-resolution algorithms

2. Sensor calibration
   - Problem formulation
   - Uniqueness
   - An optimization approach
   - Numerical simulations
Source localization with sensor array

Point sources: \( x(t) = \sum_{j=1}^{S} x_j \delta(t - \omega_j), \ \omega_j \in [0, 1) \)

Measurement at the \( m \)th sensor, \( m = 0, \ldots, M - 1 \):

\[
y_m = \sum_{j=1}^{S} x_j e^{-2\pi im\omega_j} + e_m
\]

Measurements: \( \{y_m : m = 0, \ldots, M - 1\} \)

To recover: source locations \( \{\omega_j\}_{j=1}^{S} \) and source amplitudes \( \{x_j\}_{j=1}^{S} \).
Rayleigh criterion

\[ \hat{x}(\omega) = \sum_{m=0}^{M-1} y_m \frac{e^{2\pi im\omega}}{M} \]

Rayleigh length = \(1/M\)
Inverse Fourier transform and the MUSIC algorithm

Multiple Signal Classification (MUSIC): [Schmidt 1981]

noise-free

noisy
Interesting questions

1. What is the super-resolution limit of the “best” algorithm?

2. What is the super-resolution limit of a specific algorithm?
   - MUSIC [Schmidt 1981]
   - ESPRIT [Roy and Kailath 1989]
   - the matrix pencil method [Hua and Sarkar 1990]
Existing works

1. Super-resolution limit with continuous measurements
   - Donoho 1992, Demanet and Nguyen 2015

2. Performance guarantees for well separated point sources
   - Greedy algorithms [Duarte and Baraniuk 2013, Fannjiang and L. 2012]
   - MUSIC [L. and Fannjiang 2016]
   - The matrix pencil method [Moitra 2015]

3. Performance guarantees for super-resolution
   - Total variation min for \textit{positive} sources [Morgenshtern and Candès 2016] or sources with certain sign pattern [Benedetto and Li 2016]
   - Lasso for \textit{positive} sources [Denoyelle, Duval and Peyré 2016]
Discretization on a fine grid

Point sources: \( \mu = \sum_{n=0}^{N-1} x_n \delta_{n/N} \) with \( x \in \mathbb{C}^N \)

Measurement vector

\[
y = \Phi x + e
\]

where \( \Phi \in \mathbb{C}^{M \times N} \) is the first \( M \) rows of the \( N \times N \) DFT matrix:

\[
\Phi_{m,n} = e^{-2\pi imn/N}
\]

and \( \|e\|_2 \leq \delta \).

Super-resolution factor (SRF) := \( \frac{N}{M} \)
Connection to compressive sensing

Sensing matrices contain certain rows of the DFT matrix.

(a) compressive sensing     (b) super-resolution
Min-max error

**Definition (S-min-max error)**

Fix positive integers $M, N, S$ such that $S \leq M \leq N$ and let $\delta > 0$. The $S$-min-max error is

$$
\mathcal{E}(M, N, S, \delta) = \inf_{\tilde{x} = \tilde{x}(y, M, N, S, \delta) \in \mathbb{C}^N} \sup_{x \in \mathbb{C}^N} \sup_{e \in \mathbb{C}^M: \|e\|_2 \leq \delta} \|\tilde{x} - x\|_2.
$$
Sharp bound on the min-max error

Theorem (Li and L. 2017)

There exist constants $A(S)$, $B(S)$, $C(S)$ such that:

1. **Lower bound.** If $M \geq 2S$ and $N \geq C(2S)M^{3/2}$, then

   $$\mathcal{E}(M, N, S, \delta) \geq \frac{\delta}{2B(2S)\sqrt{M}} \text{SRF}^{2S-1}.$$

2. **Upper bound.** If $M \geq 4S(2S + 1)$ and $N \geq M^2/(2S^2)$, then

   $$\mathcal{E}(M, N, S, \delta) \leq \frac{2\delta}{A(2S)\sqrt{M}} \text{SRF}^{2S-1}.$$

The best algorithm in the upper bound:

$$\min \|z\|_0 \quad \text{subject to} \|\Phi z - y\|_2 \leq \delta$$

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Multiple Signal Classification (MUSIC)

- **Pioneering work:** Prony 1795

- **MUSIC in signal processing:** Schmidt 1981

- **MUSIC in imaging:** Devaney 2000, Devaney, Marengo and Gruber 2005, Cheney 2001, Kirsch 2002

- **Related:** the linear sampling method [Cakoni, Colton and Monk 2011], factorization method [Kirsch and Grinsberg 2008]
**MUSIC**

**Assumption:** $S$ is known.

$$y_m = \sum_{j=1}^{S} x_j e^{-2\pi im\omega_j}, \quad m = 0, \ldots, M - 1.$$ 

$$H = \text{Hankel}(y) = \begin{bmatrix} y_0 & y_1 & \cdots & y_{M-L} \\ y_1 & y_2 & \cdots & y_{M-L+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{L-1} & y_L & \cdots & y_{M-1} \end{bmatrix} = \underbrace{\Phi^L}_{L \times S} \underbrace{X}_{S \times S} \underbrace{(\Phi^{M-L+1})^T}_{S \times (M-L+1)}$$

where

$$X = \text{diag}(x_1, \ldots, x_S)$$

$$\phi^L(\omega) = \begin{bmatrix} 1 & e^{-2\pi i\omega} & \cdots & e^{-2\pi i(L-1)\omega} \end{bmatrix}^T \in \mathbb{C}^L$$

$$\Phi^L = [\phi^L(\omega_1) \ldots \phi^L(\omega_S)] \in \mathbb{C}^{L \times S}.$$
MUSIC with noiseless measurements

\[ H = \Phi^L X (\Phi^{M-L+1})^T \]

Suppose \( \{\omega_j\}_{j=1}^S \) are distinct.

1. If \( L \geq S \), \( \text{rank}(\Phi^L) = S \).
2. If \( M - L + 1 \geq S \), \( \text{Range}(H) = \text{Range}(\Phi^L) \).
3. If \( L \geq S + 1 \), \( \text{rank}(\Phi^L \phi^L(\omega)) = S + 1 \) if and only if \( \omega \notin \{\omega_j\}_{j=1}^S \).

**Theorem**

If \( L \geq S + 1 \) and \( M - L + 1 \geq S \), \( \omega \in \{\omega_j\}_{j=1}^S \) iff \( \phi^L(\omega) \in \text{Range}(H) \).

Exact recovery with \( M \geq 2S \) regardless of the support.
Noise-space correlation function: $\mathcal{N}(\omega) = \frac{\|P_{\text{noise}}\phi^L(\omega)\|_2}{\|\phi^L(\omega)\|_2}$

Imaging function: $\mathcal{J}(\omega) = \frac{1}{\mathcal{N}(\omega)}$

$\mathcal{N}(\omega_j) = 0$ and $\mathcal{J}(\omega_j) = \infty$, $j = 1, \ldots, S$. 
MUSIC with noisy measurements

Three sources separated by 0.5 RL, \( e \sim N(0, \sigma^2 I_M) \)

Recall upper bound of the min-max error

\[
\mathcal{E}(M, N, S, \delta) \lesssim \frac{\delta}{\sqrt{M}} SRF^{2S-1}
\]

The noise that the “best” algorithm can handle is \( \delta \sim \left( \frac{1}{SRF} \right)^{2S-1} \).
Phase transition

- $S$ consecutive point sources on the grid with spacing $1/N$
- Support error: $d(\{\omega_j\}_{j=1}^{S}, \{\hat{\omega}_j\}_{j=1}^{S})$
- Noise $e \sim N(0, \sigma^2 I_M) + i \cdot N(0, \sigma^2 I_M)$, so $\mathbb{E}\|e\|_2 = \sqrt{2M}\sigma$.

Figure: The average $\log_2[\frac{\text{Support error}}{1/N}]$ over 100 trials with respect to $\log_{10} \frac{1}{\text{SRF}}$ (x-axis) and $\log_{10} \sigma$ (y-axis).
Super-resolution limit of MUSIC

The phase transition curve is

$$\sigma \sim \left( \frac{1}{\text{SRF}} \right)^{p(S)}$$

where

$$2S - 1 \leq p(S) \leq 2S.$$ 

Future work:

Support error by MUSC \( \lesssim \text{SRF}^{p(S)} \cdot \sigma. \)
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   - Super-resolution limit and min-max error
   - Super-resolution algorithms

2 Sensor calibration
   - Problem formulation
   - Uniqueness
   - An optimization approach
   - Numerical simulations
Sensor calibration

Measurement at the $m$-th sensor, $m = 0, \ldots, M - 1$:

$$y_m(t) = g_m \sum_{j=1}^{S} x_j(t) e^{-2\pi i m \omega_j} + e_m(t)$$

Multiple snapshots of measurements:

$$\{ y_m(t), m = 0, \ldots, M - 1, \ t \in \Gamma \}$$

To recover:

- Calibration parameters $g = \{g_m\}_{m=0}^{M-1} \in \mathbb{C}^M$
- Source locations $\{\omega_j\}_{j=1}^{S}$ and source amplitudes $x_j(t)$
Assumptions

**Matrix form:**

\[
y(t) = \text{diag}(g) \begin{pmatrix} A \\ \text{C}^{M\times M} \end{pmatrix} x(t) + e(t)
\]

\[
A_{n,j} = e^{-2\pi i m \omega_j}
\]

\[
x(t) = [x_1(t) \ldots x_S(t)]^T, \ y(t) = [y_0(t) \ldots y_{M-1}(t)]^T, \ e(t) = [e_0(t) \ldots e_{M-1}(t)]^T
\]

**Assumptions:**

1. \(|g_m| \neq 0, \ m = 0, \ldots, M - 1;\)
2. \( \mathbb{E}x(t) = 0 \) and \( \mathbb{E}e(t) = 0; \)
3. \( R^x := \mathbb{E}x(t)x^*(t) = \text{diag}(\{\gamma_j^2\}_{j=1}^S); \)
4. \( \mathbb{E}x(t)e^*(t) = 0; \)
5. \( \mathbb{E}e(t)e^*(t) = \sigma^2 I_M \) where \( \sigma \) represents noise level.
Uniqueness up to a trivial ambiguity

**Trivial ambiguity:** \{\tilde{g}, \{\tilde{\omega}_j\}_{j=1}^S, \tilde{x}(t)\} is called equivalent to \{g, \{\omega_j\}_{j=1}^S, x(t)\} up to a trivial ambiguity if there exist \(c_0 > 0, c_1, c_2 \in \mathbb{R}\):

\[
\tilde{g} = \{\tilde{g}_m = c_0 e^{i(c_1 + mc_2)} g_m\}_{m=0}^{M-1} \\
\tilde{S} = \{\tilde{\omega}_j : \tilde{\omega}_j = \omega_j - c_2/(2\pi)\}_{j=1}^S \\
\tilde{x}(t) = x(t)c_0^{-1}e^{-ic_1}.
\]

Uniqueness with infinite snapshots of noiseless measurements:

Let \(f_m = \sum_{j=1}^S \gamma_j^2 e^{2\pi i m \omega_j}\), \(m = 0, \ldots, M - 1\).

**Theorem**

Suppose \(|f_1| > 0\) and \(M \geq S + 1\). Let \{g, \{\omega_j\}_{j=1}^S, x(t)\} be a solution to the calibration problem. If there is another solution \{\tilde{g}, \{\tilde{\omega}_j\}_{j=1}^S, \tilde{x}(t)\}, then \{\tilde{g}, \{\tilde{\omega}_j\}_{j=1}^S, \tilde{x}(t)\} is equivalent to \{g, \{\omega_j\}_{j=1}^S, x(t)\}.
Covariance matrix


\[ R^y := \mathbb{E} y(t)y^*(t) = \text{diag}(g)AR^xA^* \text{diag}(\bar{g}) \]

\[ \mathcal{H} : \mathbb{C}^M \rightarrow \mathbb{C}^{M \times M} : \quad \mathcal{H}(f) := \begin{bmatrix} f_0 & \bar{f}_1 & \cdots & \bar{f}_{N-1} \\ f_1 & f_0 & \cdots & \bar{f}_{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N-1} & f_{N-2} & \cdots & f_0 \end{bmatrix} = AR^xA^*. \text{ Then} \]

\[ R^y = \text{diag}(g)\mathcal{H}(f)\text{diag}(\bar{g}) \]

\[ R^y_{m,n} = g_m\bar{g}_n f_{m-n} \]

When \( f_1 \neq 0 \), the diagonal and subdiagonal entries in \( R^y \) determine the solution up to a trivial ambiguity.
Algebraic methods

Sensitivity of the partial algebraic method:

- \( N \geq s + 1, \ |f_1| > 0 \) and sources are separated by \( 1/M \).
- Empirical covariance matrix is computed with \( L \) snapshots of measurements.

We proved that,

\[
\mathbb{E} \min_{c_0 > 0, c_1, c_2 \in \mathbb{R}} \max_m |c_0 \hat{g}_m - e^{i(c_1 + mc_2)} g_m| \leq O \left( \frac{\max(\sigma, \sigma^2)}{\sqrt{L}} \right),
\]

Partial algebraic method: only diagonal and subdiagonal entries in the covariance matrix are used.

Full algebraic method: problem of phase wrapping
An optimization approach

\[ \hat{R}^y = GAR^x A^* G^* = \text{diag}(g) \mathcal{H}(f) \text{diag}(\bar{g}) \]

Optimization problem:

\[ \min_{g, f \in \mathbb{C}^M} \mathcal{L}(g, f) := \left\| \text{diag}(g) \mathcal{H}(f) \text{diag}(\bar{g}) - \hat{R}^y \right\|_F^2. \]

- If \( \hat{R}^y = R^y \), the global minimizer of \( \mathcal{L} \) is equivalent to the ground truth \((g, f)\).
Regularized optimization

**Goal:** prevent $\|g\| \to \infty$ and $\|f\| \to 0$ (or vice versa)

$\hat{n}_0$ is an estimator of $n_0 := \|g\|^2 \|f\|$ from the partial algebraic method.

### Regularized optimization:

$$
\min_{g, f \in \mathbb{C}^N} \tilde{\mathcal{L}}(g, f) := \mathcal{L}(g, f) + \mathcal{G}(g, f)
$$

$$
\mathcal{G}(g, f) = \rho \left[ \mathcal{G}_0 \left( \frac{\|f\|^2}{2\hat{n}_0} \right) + \mathcal{G}_0 \left( \frac{\|g\|^2}{\sqrt{2\hat{n}_0}} \right) \right]
$$

where $\mathcal{G}_0(z) = (\max(z - 1, 0))^2$ and $\rho \geq \frac{3\hat{n}_0 + \|R^y - \hat{R}^y\|_F}{(\sqrt{2} - 1)^2}$

**Initialization:** $(g^0, f^0) : \|g^0\|^2 \leq \sqrt{2\hat{n}_0}, \|f^0\| \leq \sqrt{2\hat{n}_0}$

**Feasible set:** $\mathcal{N}_{\hat{n}_0} = \{(g, f) : \|g\|^2 \leq 2\sqrt{\hat{n}_0}, \|f\| \leq 2\sqrt{\hat{n}_0}\}$
Wirtinger gradient descent

for $k = 1, 2, \ldots$,

- $g^k = g^{k-1} - \eta^k \nabla g \tilde{\mathcal{L}}(g^{k-1}, f^{k-1})$
- $f^k = f^{k-1} - \eta^k \nabla f \tilde{\mathcal{L}}(g^{k-1}, f^{k-1})$

end

Theorem (Eldar, L. and Tang)

If the step length is chosen such that

$$\eta^k \leq \frac{2}{146\hat{n}_0 \max(\sqrt{n}_0, \sqrt[4]{\hat{n}_0}) + 8\hat{n}_0 + 16 \max(\sqrt{n}_0, \sqrt[4]{\hat{n}_0}) \|R^y - \hat{R}^y\|_F + \frac{8\rho}{\min(n_0, \sqrt{n}_0)}},$$

then Wirtinger gradient descent gives rise to $(g^k, f^k) \in \mathcal{N}_{\hat{n}_0}$, and

$$\|\nabla \tilde{\mathcal{L}}(g^k, f^k)\| \to 0, \text{ as } k \to \infty.$$
Sensitivity to the number of snapshots

1. the partial algebraic method
2. our optimization approach
3. an alternating minimization: [Friedlander and Weiss 1990]

- 20 sources separated by $2/M$ and noise level $\sigma = 2$

![Graph of log10(Calibration error) versus log10(#snapshot)](image1)

Relative calibration error versus $L$

![Graph of Success rate versus log10(#snapshot)](image2)

Support success rate versus $L$

**Observation:** Calibration error $= O(L^{-\frac{1}{2}})$
Sensitivity to noise level $\sigma$

- 20 sources separated by $2/M$ and $L = 500$

Relative calibration error versus $\sigma$  
Support success rate versus $\sigma$

**Observation:** Calibration error $= O(\sigma)$
Conclusion

1. Super-resolution
   - Resolution limit and a sharp bound on the min-max error
   - Resolution limit of the MUSIC algorithm

2. Sensor calibration
   - Uniqueness with infinite snapshots of noiseless data
   - The partial algebraic method and a stability analysis
   - An optimization approach and convergence to a stationary point
Thank you for your attention!

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